Economic fluctuations and anomalous diffusion

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We quantify the relation between trading activity — measured by the number of transactions $N_{\Delta t}$—and the price change $G_{\Delta t}$ for a given stock, over a time interval $[t, t+\Delta t]$. To this end, we analyze a database documenting every transaction for 1000 U.S. stocks for the two-year period 1994–1995. We find that price movements are equivalent to a complex variant of classic diffusion, where the diffusion constant fluctuates drastically in time. We relate the analog for stock price fluctuations of the diffusion constant—known in economics as the volatility—to two microscopic quantities: (i) the number of transactions $N_{\Delta t}$ in Δt, which is the analog of the number of collisions and (ii) the variance $W^2_{\Delta t}$ of the price changes for all transactions in Δt, which is the analog of the local mean square displacement between collisions. Our results are consistent with the interpretation that the power-law tails of $P(G_{\Delta t})$ are due to $P(W_{\Delta t})$, and the long-range correlations in $|G_{\Delta t}|$ are due to $N_{\Delta t}$.

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Consider the diffusion [1] of an ink particle in water. Starting out from a point, the ink particle undergoes a random walk due to collisions with the water molecules. The distance covered by the particle after a time Δt is

$$X_{\Delta t}=\sum_{i=1}^{N_{\Delta t}} \delta x_i , \tag{1a}$$

where δx_i are the distances that the particle moves in between collisions, and $N_{\Delta t}$ denotes the number of collisions during the interval Δt. The distribution $P(X_{\Delta t})$ is Gaussian with a variance $\langle X^2_{\Delta t}\rangle = N_{\Delta t} W^2_{\Delta t} \equiv D\Delta t$, where the local mean square displacement $W^2_{\Delta t}\equiv\langle(\delta x_i)^2\rangle$ is the variance of the individual steps δx_i in the interval $[t, t+\Delta t]$, and D is the diffusion constant.

For the classic diffusion problem considered above, (i) the probability distribution $P(N_{\Delta t})$ is a ''narrow'' Gaussian, i.e., has a standard deviation much smaller than the mean $\langle N_{\Delta t}\rangle$, and Δt is such that $\langle N_{\Delta t}\rangle$ is sufficiently large, (ii) the time between collisions of an ink particle are not strongly correlated, so $N_{\Delta t}$ at any future time $t+\tau$ depends at most weakly on $N_{\Delta t}$ at time t—i.e., the correlation function $\langle N_{\Delta t}(t) N_{\Delta t}(t+\tau)\rangle$ has a short-range exponential decay, (iii) the distribution $P(W^2_{\Delta t})$ is also a narrow Gaussian, (iv) the correlation function $\langle W_{\Delta t}(t) W_{\Delta t}(t+\tau)\rangle$ has a short-range exponential decay, and (v) the variable $\epsilon\equiv X_{\Delta t}/(W_{\Delta t}\sqrt{N_{\Delta t}})$ is uncorrelated and Gaussian distributed. These conditions imply that $X_{\Delta t}$ is Gaussian distributed and short-range correlated.

An ink particle diffusing under more general conditions—such as in a bubbling hot spring, where the characteristics of bubbling depend on a wide range of time and length scales—would result in a quite different distribution of $X_{\Delta t}$. In the following, we will present empirical evidence that the movement of stock prices is equivalent to a complex variant of classic diffusion, specified by the following conditions: (i) $P(N_{\Delta t})$ is not a Gaussian, but has a power-law tail, (ii) $N_{\Delta t}$ has long-range power-law time-correlations, (iii) $P(W^2_{\Delta t})$ is not a Gaussian, but has a power-law tail, (iv) the correlation function $\langle W_{\Delta t}(t) W_{\Delta t}(t+\tau)\rangle$ is short ranged, and (v) the variable $\epsilon\equiv X_{\Delta t}/(W_{\Delta t}\sqrt{N_{\Delta t}})$ is Gaussian distributed and short-range correlated. Under these conditions, the statistical properties of $X_{\Delta t}$ will depend on the exponents characterizing these power laws.

Just as the displacement $X_{\Delta t}$ of a diffusing ink particle is the sum of $N_{\Delta t}$ individual displacements δx_i, so also the stock price change $G_{\Delta t}$ is the sum of the price changes δp_i of the $N_{\Delta t}$ transactions in the interval $[t,t+\Delta t]$,

$$G_{\Delta t}=\sum_{i=1}^{N_{\Delta t}} \delta p_i . \tag{1b}$$

Figure 1(a) shows $N_{\Delta t}$ for classic diffusion and for one stock (Exxon Corporation). The number of trades for Exxon displays several events the size of tens of standard deviations and hence is inconsistent with a Gaussian process [2].

(i) We first analyze the distribution of $N_{\Delta t}$ [Fig. 1(b)]. Figure 1(c) shows that the cumulative distribution of $N_{\Delta t}$ displays a power-law behavior $P\{N_{\Delta t}>x\}\sim x^{-\beta}$. For the 1000 stocks analyzed [3], we obtain a mean value $\beta=3.40\pm0.05$. Note that $\beta>2$ is outside the Lévy stable domain $0<\beta<2$.

(ii) We next determine the correlations in $N_{\Delta t}$. We find that the correlation function $\langle N_{\Delta t}(t) N_{\Delta t}(t+\tau)\rangle$ is not exponentially decaying as in the case of classic diffusion, but rather displays a power-law decay [Fig. 1(d)]. This result quantifies the qualitative fact that if the trading activity ($N_{\Delta t}$) is large at any time, it is likely to remain so for a considerable time thereafter.

(iii) We then compute the variance $W^2_{\Delta t}\equiv\langle(\delta p_i)^2\rangle$ of the individual changes δp_i due to the $N_{\Delta t}$ transactions in the interval $[t, t+\Delta t]$, Fig. 2(a). We find that the distribution $P(W_{\Delta t})$ displays a power-law decay $P\{W_{\Delta t}>x\}\sim x^{-\gamma}$

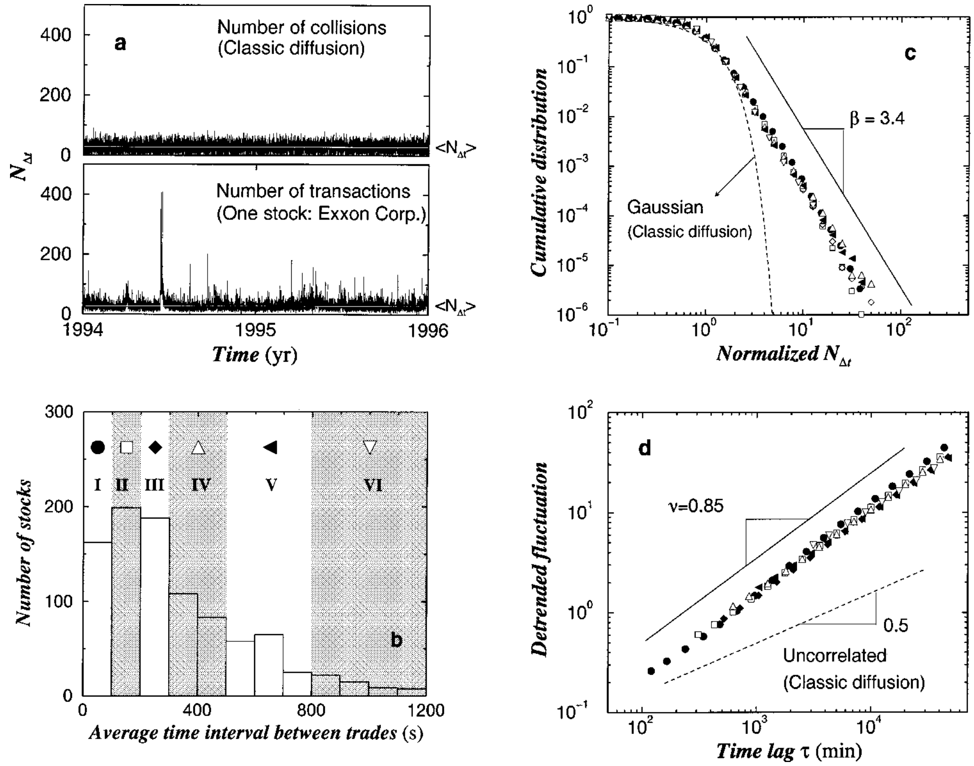

FIG. 1. (a) The lower panel shows $N_{\Delta t}$ for Exxon Corporation with $\Delta t = 15$ min. The upper panel shows a sequence of uncorrelated Gaussian random numbers with the same mean $\langle N_{\Delta t}\rangle \approx 28$ and standard deviation (≈ 16), which depicts the number of collisions $N_{\Delta t}$ for the classic diffusion problem. Note that in contrast to diffusion, $N_{\Delta t}$ for Exxon shows frequent large events of the magnitude of tens of standard deviations, which would be ''forbidden'' for Gaussian statistics. (b) Histogram of the average time interval between trades $\langle \delta t\rangle$ for the 1000 stocks studied. In order to ensure that the sampling time interval Δt for each stock contains a sufficient number of transactions, we partition the stocks into six groups (I–VI) based on $\langle \delta t\rangle$. Each group contains approximately 150 stocks. For a specific group, we choose a sampling time Δt at least 10 times larger than the average value of $\langle \delta t\rangle$ for that group. We choose the sampling time interval $\Delta t = 15, 39, 65, 78, 130$, and 390 min, respectively, for groups I–VI. (c) Log-log plot of the cumulative distribution of $N_{\Delta t}$ for the stocks in each of the six groups in (b). Since each stock has a different average value of $\langle N_{\Delta t}\rangle$, we use a normalized number of transactions $n_{\Delta t} \equiv N_{\Delta t}/\langle N_{\Delta t}\rangle$. Each symbol shows the cumulative distribution $P\{n_{\Delta t} > x\}$ of the normalized number of transactions $n_{\Delta t}$ for all stocks in each group. An analysis of the exponents obtained by fits to the cumulative distributions $P\{N_{\Delta t} > x\}$ of each of the 1000 stocks yields an average value $\beta = 3.40 \pm 0.05$. (d) In order to accurately quantify power-law time correlations in $N_{\Delta t}$, we use the method of detrended fluctuations [9]. We plot the detrended fluctuations $F(\tau)$—defined as the root-mean-square deviation of the integrated signal around a linear fit in a window of length τ—as a function of the time scale τ, for each of the six groups. Absence of long-range correlations would imply $F(\tau) \sim \tau^{0.5}$, whereas $F(\tau) \sim \tau^{\nu}$ with $0.5 < \nu \leqslant 1$ shows a power-law decay of the correlation function with exponent $\nu_c = 2 - 2\nu$. For each group, we plot $F(\tau)$ averaged over all stocks in that group. In order to detect genuine long-range correlations, the U-shaped intraday pattern for $N_{\Delta t}$ has been removed by dividing each $N_{\Delta t}$ by the intraday pattern [9]. We obtain the mean value $\nu = 0.85 \pm 0.01$ from the exponents ν obtained by power-law fits to $F(\tau)$ for each of the 1000 stocks.

[Fig. 2(b)]. For the 1000 stocks analyzed, we obtain a mean value of the exponent $\gamma = 2.9 \pm 0.1$.

(iv) Next, we quantify correlations in $W_{\Delta t}$. We find that the correlation function $\langle W_{\Delta t}(t)\, W_{\Delta t}(t+\tau)\rangle$ shows only weak correlations [Fig. 2(c)]. This means that $W_{\Delta t}$ at any future time $t+\tau$ depends at most weakly on $W_{\Delta t}$ at time t.

(v) Finally, we consider a statistical property that is ''local'' in time. Suppose δp_i are chosen *only from the interval* $[t, t+\Delta t]$, and let us hypothesize that *these* δp_i are mutually independent, with a common distribution $P(\delta p_i | t \in [t, t+\Delta t])$ having a finite variance $W^2_{\Delta t}$. Under this hypothesis, the central limit theorem, applied to the sum of δp_i in Eq. (1b), implies that the ratio

$$\epsilon \equiv \frac{G_{\Delta t}}{W_{\Delta t}\sqrt{N_{\Delta t}}} \tag{2}$$

must be a Gaussian-distributed random variable with zero mean and unit variance [4]. Indeed, for classic diffusion, $X_{\Delta t}/(W_{\Delta t}\sqrt{N_{\Delta}})$ is Gaussian-distributed and uncorrelated [Fig. 3(a)]. We confirm the validity of this hypothesis by analyzing (i) the distribution $P(\epsilon)$, which we find to be consistent with Gaussian behavior [Fig. 3(b)], and (ii) the correlation function $\langle \epsilon(t)\, \epsilon(t+\tau)\rangle$, for which we find only short-range correlations [Fig. 3(c)].

Thus far, we have seen that the data for stock price movements support the following results: (i) the distribution of $N_{\Delta t}$ decays as a power law, (ii) $N_{\Delta t}$ has long-range correlations, (iii) the distribution of $W_{\Delta t}$ decays as a power law, (iv) $W_{\Delta t}$ displays only weak correlations, and (v) the variable $\epsilon \equiv G_{\Delta t}/(W_{\Delta t}\sqrt{N_{\Delta t}})$ is Gaussian-distributed and uncorrelated, i.e., the price change $G_{\Delta t}$ *at any time* (for a given value of $N_{\Delta t}$ and $W_{\Delta t}$) is consistent with a Gaussian-distributed random variable [2,5] with an ''instantaneous'' variance

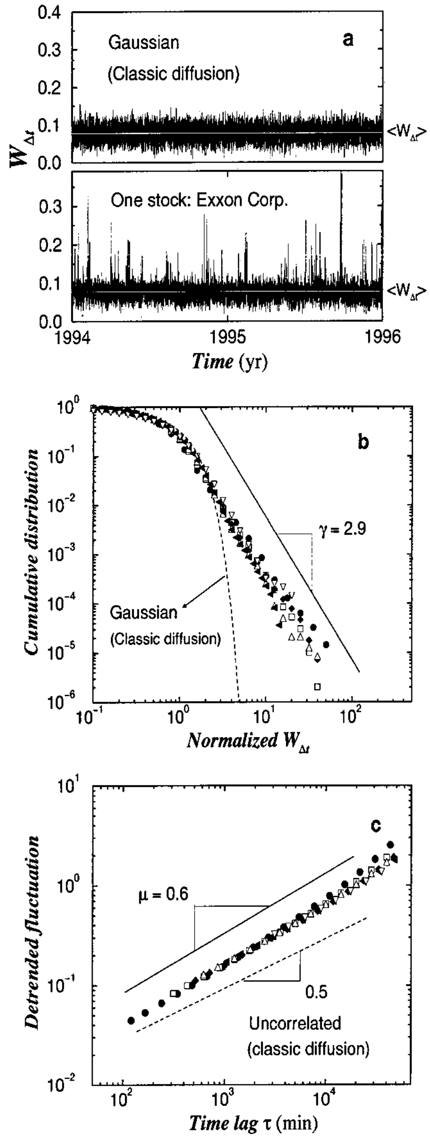

FIG. 2. (a) Standard deviation $W_{\Delta t}$ computed from price changes δp_i due to every transaction in the interval $[t, t+\Delta t]$ for Exxon Corporation for $\Delta t = 15$ min (lower panel) in contrast to uncorrelated Gaussian random numbers with the same mean value $\langle W_{\Delta t}\rangle \approx 0.08$ and variance (upper panel). Intervals having fewer than ten transactions are not used for computing $W_{\Delta t}$. The time series of $W_{\Delta t}$ for Exxon shows a number of large events of the size of tens of standard deviations. Note that the large values of $N_{\Delta t}$ in Fig. 1(a) do not correspond to large values of $W_{\Delta t}$, showing that $N_{\Delta t}$ and $W_{\Delta t}$ are only weakly correlated (we find $\langle N_{\Delta t} W_{\Delta t}\rangle = 0.16$ for the 1000 stocks studied). (b) Log-log plot of the cumulative distribution of $W_{\Delta t}$ for each of the six groups defined in Fig. 1(b). Since the average value $\langle W_{\Delta t}\rangle$ changes from one stock to another, we normalize $W_{\Delta t}$ by $\langle W_{\Delta t}\rangle$. Each symbol shows the cumulative distribution of the normalized $W_{\Delta t}$ for all stocks in each group. We analyze the power-law exponents γ obtained by fits to the cumulative distributions of $W_{\Delta t}$ of each of the 1000 stocks separately, and find an average value $\gamma = 2.9 \pm 0.1$. (c) Log-log plot of the detrended fluctuation $F(\tau)$ for the six groups as a function of the time lag τ. We calculate the detrended fluctuation exponents by fitting $F(\tau)$ for each stock separately and find an average value $\mu = 0.60 \pm 0.01$, significantly smaller than our result $\nu = 0.85 \pm 0.01$ for $N_{\Delta t}$. The same procedure for $V_{\Delta t}$ yields the value 0.80 ± 0.01 for its detrended fluctuation exponent, consistent with the value of $\nu = 0.85 \pm 0.01$ for $N_{\Delta t}$.

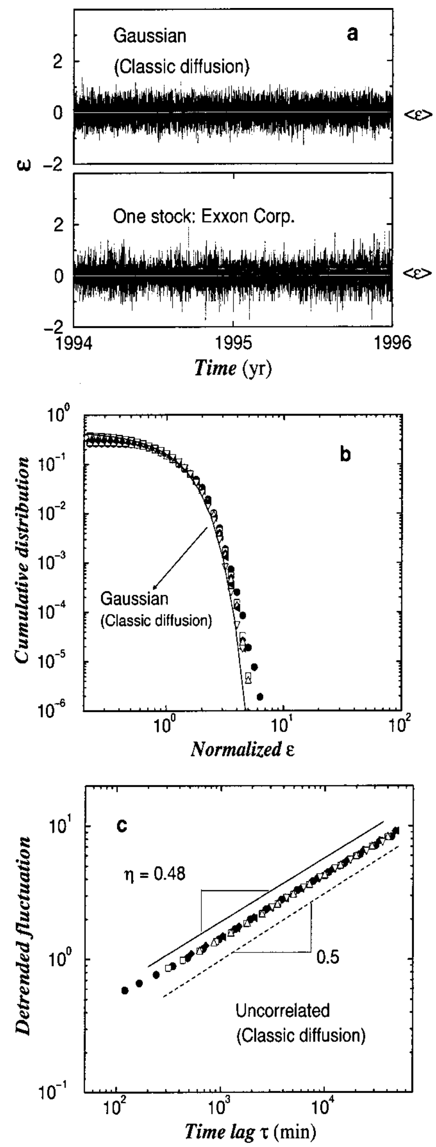

FIG. 3. (a) Time series of $\epsilon \equiv G_{\Delta t}/(W_{\Delta t}\sqrt{N_{\Delta t}})$ for Exxon Corporation for $\Delta t = 15$ min (lower panel) contrasted with a sequence of uncorrelated Gaussian random numbers with the same mean and variance, which depicts $X_{\Delta t}/(W_{\Delta t}\sqrt{N_{\Delta t}})$ for classic diffusion (upper panel). (b) Positive tail of the cumulative distribution of ϵ for the six groups. We normalize ϵ by its standard deviation in order to compare different stocks. Each symbol shows the cumulative distributions of the normalized ϵ for all stocks in each of the six groups. The negative tail (not shown) displays similar behavior. For the 1000 stocks studied, we obtain the average value of kurtosis 3.46 ± 0.03 and skewness 0.018 ± 0.002. (c) Log-log plot of the detrended fluctuation $F(\tau)$ averaged for all stocks belonging to each of the six groups. We obtain the mean value $\eta = 0.48 \pm 0.01$ from the detrended fluctuation exponents obtained by power-law fits to $F(\tau)$ for each stock.

$N_{\Delta t} W_{\Delta t}^2$.

Next, we explore the implications of our empirical findings. Namely, we show how the statistical properties [6–9] of price changes $G_{\Delta t}$ can be understood in terms of the properties of $N_{\Delta t}$ and $W_{\Delta t}$. We will argue that the pronounced tails of the distribution of price changes [6,7] are largely due to $W_{\Delta t}$ and the long-range correlations of the volatility [8,9] are largely due to the long-range correlations in $N_{\Delta t}$. By contrast, in classic diffusion $N_{\Delta t}$ and $W_{\Delta t}$ do not change the Gaussian behavior of $X_{\Delta t}$ because they have only uncorrelated Gaussian fluctuations [2,4].

Consider first the distribution of price changes $G_{\Delta t}$, which decays as a power law $P\{G_{\Delta t}>x\}\sim x^{-\alpha}$ with an exponent $\alpha\approx 3$ [7]. Above, we reported that the distribution $P\{N_{\Delta t}>x\}\sim x^{-\beta}$ with $\beta\approx 3.4$ [Fig. 1(c)]. Therefore, $P\{\sqrt{N_{\Delta t}}>x\}\sim x^{-2\beta}$ with $2\beta\approx 6.8$. Equation (2) then implies that $N_{\Delta t}$ alone cannot explain the value $\alpha\approx 3$. Instead, $\alpha\approx 3$ must arise from the distribution of $W_{\Delta t}$, which indeed decays with approximately the same exponent $\gamma\approx\alpha\approx 3$ [Fig. 2(b)]. Thus the power-law tails of $P(G_{\Delta t})$ originate from the power-law tail of $P(W_{\Delta t})$.

Next, consider the long-range correlations found for the volatility $V_{\Delta t}$ [8,9], which is the analog of the diffusion constant *D*. Just as in classic diffusion, where the diffusion constant *D* is related to the variance of $X_{\Delta t}$ through the relation $D\Delta t \equiv \langle X_{\Delta t}^2\rangle = N_{\Delta t} W_{\Delta t}^2$, so the volatility is defined as the ''local'' standard deviation of $G_{\Delta t}$ through the relation $V_{\Delta t}^2 \equiv \langle G_{\Delta t}^2\rangle = N_{\Delta t} W_{\Delta t}^2$. Above, we reported that the number of transactions $N_{\Delta t}$ displays long-range correlations, whereas $W_{\Delta t}$ displays only weak correlations (Figs. 1 and 2). Therefore, the long-range correlations in $V_{\Delta t}$ should arise from those found in $N_{\Delta t}$. Indeed, we find that the correlations in $V_{\Delta t}$ decay as a power law with an exponent similar to that of $N_{\Delta t}$. Hence, while the power-law tails in $P(G_{\Delta t})$ are due to the power-law tails in $P(W_{\Delta t})$, the long-range correlations of $V_{\Delta t}$ are due to those of $N_{\Delta t}$.

In summary, we have found that stock price movements are analogous to a complex variant of classic diffusion, where the analog of the diffusion constant fluctuates drastically in time. Furthermore, we have quantified and empirically demonstrated the relation between stock price changes and trading activity. The implications of our results for the number of transactions $N_{\Delta t}$ and the local standard deviation $W_{\Delta t}$ are of potential interest. The fluctuations in $N_{\Delta t}$ reflect the trading activity for a given stock and its power-law distribution and long-range correlations may be related to ''avalanches,'' where trades beget new trades [10]. The fluctuations in $W_{\Delta t}$ reflect several factors: (i) the level of liquidity of the market, (ii) the risk aversion of the market participants, and (iii) the uncertainty about the fundamental value of the asset.

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