Cooperation in scale-free networks with limited associative capacities

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In this work we study the effect of limiting the number of interactions (the associative capacity) that a node can establish per round of a prisoner's dilemma game. We focus on the way this limitation influences the level of cooperation sustained by scale-free networks. We show that when the game includes cooperation costs, limiting the associative capacity of nodes to a fixed quantity renders in some cases larger values of cooperation than in the unrestricted scenario. This allows one to define an optimum capacity for which cooperation is maximally enhanced. Finally, for the case without cooperation costs, we find that even a tight limitation of the associative capacity of nodes yields the same levels of cooperation as in the original network.

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Recent studies have shown that cooperation among unrelated individuals has a much better chance to survive when their interaction patterns are described through a complex network [1–7]. Specifically, it has been proven that the heterogeneous character of the number of social contacts each individual has not only reproduces the topological features of many social systems [8,9] much better, but also greatly favors the emergence of cooperation. The paradigmatic example of such enhancement of cooperation appears within the framework of the prisoner's dilemma (PD) game on scale-free (SF) networks. In these structures, the probability of finding an individual with k social contacts is given by a power-law distribution $P(k) \sim k^{-\gamma}$. Therefore, although most individuals have few connections, there exists a significant number of individuals, the hubs, with a large number of social contacts. The special topology of SF networks favors the formation of cooperator clusters centered around the hubs of the network. These clusters of cooperators provide their members with a stable source of benefits, which ultimately allows them to resist cycles of invasions from defectors [10].

The ability of SF networks to promote cooperation has been intensively studied in recent years by incorporating other mechanisms and topological features that could further enhance or decrease the emergence of cooperation [11,12]. Studies about the topological impact of the SF architecture include the effects that the average connectivity [13], the clustering coefficient [14,15], and the degree-degree correlations [16] have on the evolutionary success of cooperation. In addition, different mechanisms such as reputation [17,18], the diversity of reproductive rates [19], and the coevolution of different update rules [20] have shown how to reinforce the resilience of cooperation in SF graphs that has turned out to be robust under different scenarios [21,22]. However, the amplification of cooperation provided by SF networks decreases when other features such as the normalization of payoffs to the degree of the nodes [23,24], the conformity of players, [25], or other update rules [26,27] are introduced.

The above works aim at modeling relevant aspects of human behavior in social networks. Nonetheless, it is usually assumed that the number of interactions that a node establishes in every round of an evolutionary game is equal to the number of topological neighbors it has, as dictated by the complex network of interactions. This consideration, although valid when dealing with a regular network whose nodes have a moderate degree of interaction, is difficult to justify when heterogeneous networks are considered. In particular, it is hard to sustain that a social hub will interact concurrently with all its acquaintances during the time window associated with a round robin of the evolutionary game. In other words, one should abandon the hypothesis of an unlimited associative capacity of the individuals. Admittedly, this restriction has been previously found to play a key role in the spreading of diseases [28,29].

In this work we are interested in studying the effect of restricting the number of interactions that a node is allowed to make per unit of time when involved in a social dilemma on SF networks. To this end, we consider that every node of the network will only establish a fixed number k^* of interactions randomly chosen among its topological neighbors. The specific social dilemma that will take place on top of the SF networks will be a PD game, and we will consider its formulation both with and without cost per cooperation. Moreover, we will also consider two types of updating rules: a replicatorlike rule and the Fermi-like setup. Our results indicate that in the case of a cost-per-cooperation formulation of the prisoner's dilemma game, and for both update rules, there is an optimum number of interactions k^*_{opt} which renders larger values of cooperation in the system than the (usual) unlimited scenario. Nevertheless, when no cost per cooperation is considered, we find that the level of cooperation increases with k^*. However, this growing behavior saturates at values of k^* well below its maximum possible value, thus pointing out that it is not necessary to exploit the full associative capacity of network nodes to achieve the large levels of cooperation observed in SF networks.

We first build up scale-free networks using the Barabási-Albert (BA) procedure [30]. Starting from a small set m_0 of fully connected nodes, we sequentially add a new node j to the network. Every new node will attach to m of the existing nodes. The probability that a link from j connects with an existing node i is proportional to its degree, $P_i = \frac{k_i}{\sum_l k_l}$. This procedure continues until the network reaches its final size N.

The degree distribution, i.e., the probability of finding a node in the network with k neighbors, is a power-law $P(k) \sim k^{-\gamma}$ with an exponent $\gamma = 3$, and the average connectivity is $\langle k \rangle = 2m$. In our case, we have used networks of size $N = 4 \times 10^3$ nodes and an average value for the connectivity $\langle k \rangle = 4$.

We consider that every node on the network represents a player of a prisoner's dilemma (PD) game so that it can adopt two possible strategies: cooperation (C) and defection (D). The initial strategies of the players are randomly assigned with equal probability. As usual in an evolutionary setting, we iterate a large number of rounds of the PD game. However, in each of these rounds, we do not let nodes play with all their topological neighbors, but we force each node to choose, also randomly, k^* partners among their respective acquaintances. Obviously, when a node has $k_i < k^*$, it will play with all its neighbors in every round. However, when $k_i > k^*$, a node will choose a subset of them, making a different selection in every round. Notice that in order to preserve the symmetry of the interactions, when a node i chooses a node j, it automatically implies that j also plays with i (even in the case where j does not choose i to play in this round). Therefore, once a node has chosen its k^* partners, the total number of nodes that play with it (its effective connectivity) is not strictly k^* but in general $k_i^{\text{eff}} > k^*$. It is easy to show that for a SF network, like those used in this work, this effective connectivity reads

$$k_i^{\text{eff}} \approx k^* + k_i \left[1 - \frac{k^*}{k_i}\right]\left[1 - \frac{1}{k^*}\right]. \tag{1}$$

The above expression implies that the average effective degree $\langle k^{\text{eff}} \rangle$ of the interaction network relates with the topological one $\langle k \rangle$ as

$$\langle k^{\text{eff}} \rangle \approx \langle k \rangle \left[1 - \left(\frac{\langle k \rangle}{2k^*}\right)^2\right]. \tag{2}$$

Once all of the nodes have set their current effective neighborhood, they play a PD game with each of their k_i^{eff} game mates. We first explore the interaction between cooperators and defectors by means of a PD with participation cost. In this setting, a cooperator node i pays a cost c for each of the k_i^{eff} games, while the node playing with it obtains a payoff b (being $b > c$). However, nodes playing as defectors pay no cost and distribute no benefits. Under these conditions, the payoff matrix associated with each game reads [31]

$$\begin{array}{cc} & \begin{array}{cc} C & D \end{array} \\ \begin{array}{c} C \\ D \end{array} & \begin{pmatrix} b-c & -c \\ b & 0 \end{pmatrix} \end{array} \sim \begin{array}{cc} & \begin{array}{cc} C & D \end{array} \\ \begin{array}{c} C \\ D \end{array} & \begin{pmatrix} b/c-1 & -1 \\ b/c & 0 \end{pmatrix} \end{array}. \tag{3}$$

Note that the second payoff matrix defines an equivalent PD game with the advantage of having a single parameter, which is the benefit-to-cost ratio b/c. Obviously, the larger the ratio b/c gets, the cheaper it becomes to be a cooperator.

The benefits collected by a node i after playing with its effective neighborhood are finally accumulated and constitute its evolutionary fitness π_i. Immediately afterwards, each node updates its strategy by comparing its own payoff with the payoff of one of its neighbors randomly chosen from the current effective neighborhood. For the probability that a node i imitates the strategy of the chosen neighbor j for the next round of the game, we use the so-called Fermi rule [32–34],

$$P_{i \to j} = \frac{1}{1 + e^{w(\pi_i - \pi_j)}}, \tag{4}$$

where w is a parameter that accounts for the importance of the relative difference of payoffs on the change of strategy of node i. Notice that for $w \to \infty$, $P_{i\to j}$ depends strongly on the payoff difference, so that if $\pi_i < \pi_j$, then i will almost surely imitate j, while if $\pi_i > \pi_j$, then i will not take the strategy of j in most cases. However, for $w \to 0$, the probability of changing strategies is $P_{i\to j} \simeq 1/2$, regardless of the values of the payoffs (this case is referred to as the random drift limit). The results shown in this work correspond to the value $w = 1$. Nonetheless, we have checked that they are quite robust, as other values of w give qualitatively the same outcomes.

We iterate the above discrete-time dynamics for a large number of time steps, until the system reaches one of the two absorbing states: fixation of cooperation (all-C state) or fixation of defection (all-D state). The dynamics always ends up in one of the two latter frozen equilibria due to the irrational changes of strategies allowed by the Fermi rule given by Eq. (4), i.e., a node always have a nonzero probability of adopting the neighbor's strategy, even when the neighbor's payoff is smaller than its own. Therefore, it is necessary to make a large number of different realizations of the network dynamics for a particular value of the game parameter b/c so as to compute the probability of fixation to the all-C state $\langle c \rangle$. In this way, the outcome of the evolutionary dynamics is described by the function $\langle c \rangle(b/c)$.

In Fig. 1 we show $\langle c \rangle$ as a function of the ratio b/c for different values of the restriction k^*. As expected, the larger the value of b/c is, the cheaper being a cooperator is, and thus the larger the probability of fixation to cooperation on the system is. However, we have found a stunning and nontrivial dependence with the value of the restriction for the number of connections k^*. From Fig. 1 it is clear that for low values of b/c, i.e., when the cooperation is relatively expensive, the largest level of $\langle c \rangle$ is achieved when no restriction is imposed to the connectivity of the nodes ($k^* = k_{\max}$ actually means that every node i plays always with all its k_i topological neighbors). However, for larger values of the ratio b/c, the opposite trend occurs, and a network with some level of connectivity restriction performs better than the original one, meaning that it achieves larger values of $\langle c \rangle$. Moreover, our numerics have shown that a too restrictive value for $k^* \lesssim 10$ always performs worse, regardless of the value of b/c.

The observed crossover occurring for intermediate values of b/c suggests that for each value of b/c in this range, there exists an optimum value of k^* for which the maximum value of $\langle c \rangle$ is reached. In Fig. 2 we represent the probability of fixation to cooperation as a function of k^* for a fixed value of the ratio b/c. As predicted, we obtain a nonmonotonous behavior of $\langle c \rangle$ as k^* increases, yielding an optimum value of k^*, namely, k^*_{opt}. From the different curves, we observe that the value of k^*_{opt} becomes larger as cooperation gets more expensive, as expected from Fig. 1. On more general grounds, the optimal cooperation reached at k^*_{opt} points out a resonancelike phenomenon, such as those found in [35–37] by tuning the game parameters.

In order to better understand the origin of the optimum value for the number of interactions k^*_{opt}, we next study the

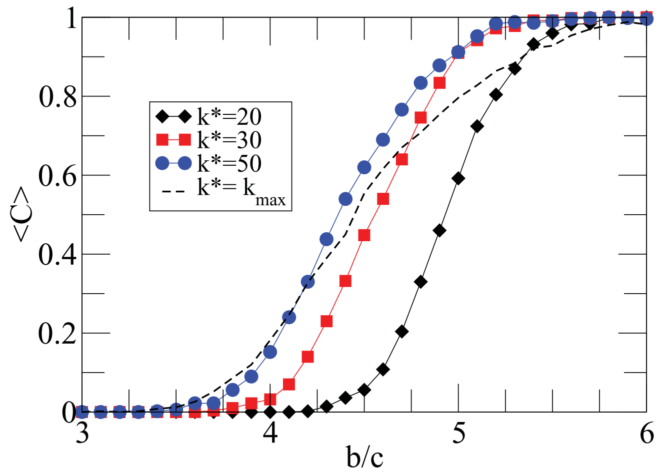

FIG. 1. (Color online) Probability of fixation to cooperation $\langle c\rangle$ as a function of the ratio b/c for a PD game with limited associative capacities using a Fermi updating rule. The scale-free networks are made up of $N = 4 \times 10^3$ nodes, and the average connectivity is $\langle k\rangle = 4$. Every point is the average over 500 different realizations.

evolution of cooperation in a different scenario: the PD game without cooperation costs. By introducing this change in our original model, we want to unveil the role of cooperation costs in the observed optimum in the number of interactions per node. In this way, we will consider the PD game together with the Fermi update given by Eq. (4), with the payoff matrix as

$$\begin{array}{cc} & \begin{array}{cc} C & D \end{array} \\ \begin{array}{c} C \\ D \end{array} & \begin{pmatrix} R & S \\ T & P \end{pmatrix} \end{array} = \begin{array}{cc} & \begin{array}{cc} C & D \end{array} \\ \begin{array}{c} C \\ D \end{array} & \begin{pmatrix} 1 & 0 \\ b & 0 \end{pmatrix} \end{array}, \tag{5}$$

where we have fixed, as usual, $R = 1$ and $P = S = 0$, while the temptation to defect, $T = b$, remains as the only parameter of the evolutionary dynamics.

In Fig. 3 we show the probability of fixation to cooperation in the system as a function of the restriction k^* for different

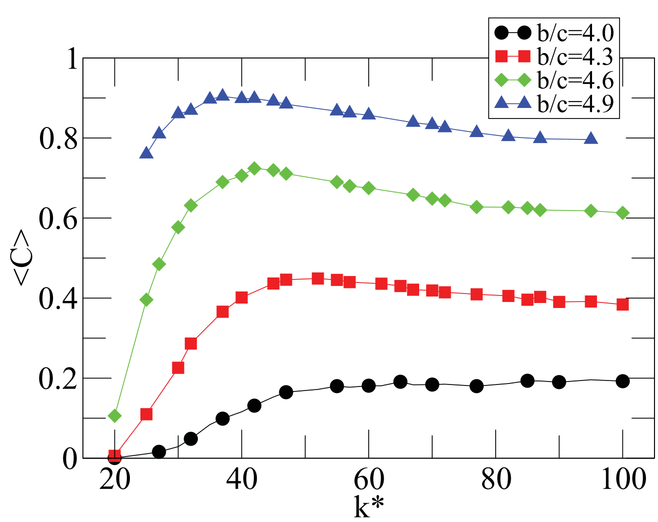

FIG. 2. (Color online) Probability of fixation to cooperation $\langle c\rangle$ as a function of the restriction k^* for different values of the ratio b/c. Network parameters are those used in Fig. 1. Every point is the average over 2×10^3 different realizations.

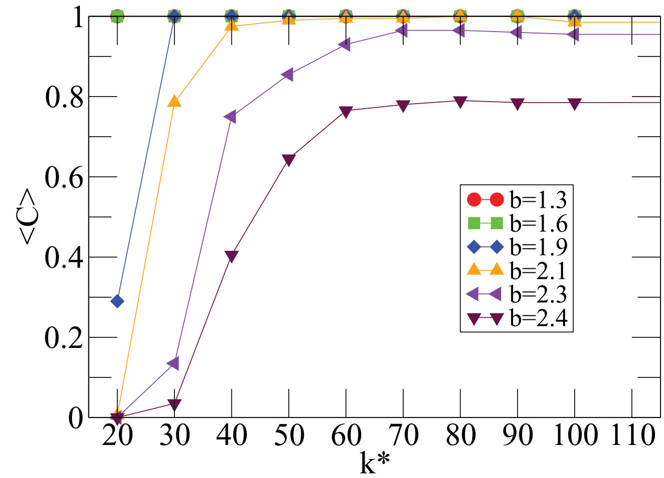

FIG. 3. (Color online) Probability of fixation to cooperation $\langle c\rangle$ as a function of the associative capacity k^* for the case of a PD game without cooperation cost combined with the Fermi updating rule. Network parameters are those used in Fig. 1. Every point is the average over 200 different realizations.

values of b. We clearly observe that for any fixed value of b, the value of $\langle c\rangle$ grows with k^*. A direct comparison between Fig. 3 and Fig. 2 suggests that the optimum value k^*_{opt} was previously observed due to the compromise for each node between the costs associated to cooperate with all its neighbors and the benefits obtained in those interactions. In fact, when cooperation costs are applied, a cooperator within a neighborhood containing a given number of defector neighbors decreases its payoff as its associative capacity increases. However, in the absence of costs, the benefits received by this cooperator do not decrease (on average) as k^* increases. Furthermore, although an optimal value k^*_{opt} is not observed, all the curves $\langle c\rangle(k^*)$ saturate beyond a certain value of $k^* = k'$ that is well below the largest degree of the network. Therefore, from this value on, increasing further the number of interacting partners does not benefit cooperation. In this sense, the value k' points out that it is not necessary to exploit the full associative capacity of nodes to achieve large values for the fixation to cooperation.

Finally, as a further check, let us consider again the PD game with cooperation costs, given by Eq. (3), combined with the replicatorlike update rule [3]. Following this update rule, after each round of the PD game, each individual i chooses at random one member j of its effective neighborhood and compares their payoffs. If $\pi_i > \pi_j$, then nothing happens (i keeps playing with the same strategy), but if $\pi_j > \pi_i$, then i will imitate the probability of j with probability

$$P_{i\to j} = \frac{\pi_j - \pi_i}{\max\left(k_i^{\mathrm{eff}}, k_j^{\mathrm{eff}}\right) b}. \tag{6}$$

Note that at variance with the Fermi rule, in this case a node will not imitate a worse performing strategy. As a consequence, the computed values of $\langle c\rangle$ account for the average fraction of cooperators in the dynamical equilibrium. In Fig. 4 we show the curves $\langle c\rangle(k^*)$ for different values of b/c. In this setting, the optimum capacity k^*_{opt} reappears, although it is not as pronounced as in the formulation using a

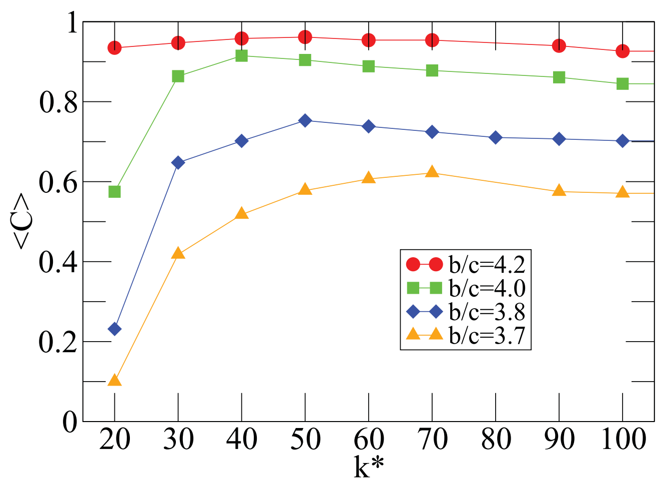

FIG. 4. (Color online) Average level of cooperation $\langle c \rangle$ as a function of the mate limitation k^* using the replicator updating rule and the formulation of the prisoner's dilemma with cost for cooperation. Network parameters are those used in Fig. 1. Every point is the average over at least 200 different realizations.

Fermi update rule. However, this latter result further confirms that the key ingredient for the existence of an optimum value for the associative capacity is the cost associated to cooperation.

In summary, in this Brief Report we have studied a realistic scenario where the number of interactions that a node can establish per round of the game is restricted to a maximum value k^*, regardless of the topological connectivity of the nodes. We have studied the effect of such restrictions on the cooperation achieved by SF networks playing a PD game with cooperation costs. Our main result is that for a range of values of the benefit-to-cost ratio b/c of the payoff matrix, the largest probability of fixation to cooperation is achieved when the associative capacity of nodes is limited, i.e., the higher levels of cooperative behavior do not occur for the original SF network. This result allows us to define an optimal associative capacity $k^*_{\rm opt}$ for a fixed value of the ratio b/c. The optimum is related to the tradeoff between the cost of cooperating and the number of neighbors a node plays with. This hypothesis is confirmed by changing the formulation of the social dilemma to the one in which the cooperation cost is zero. We have shown that, in most cases, a moderate limitation of the associative capacity of nodes leads to the same degree of cooperation as in the original (unlimited) network.

In conclusion, the results shown in this work point out that although the degree of heterogeneity of SF networks does greatly favor cooperation, it is possible to obtain larger (PD with cooperation costs) or similar (PD without cooperation costs) levels of cooperation by limiting the associative capacities of the nodes. Therefore, it is not necessary to exploit the full associative capacity of nodes in SF networks to reach large levels of cooperation.

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