Universality classes for rice-pile models

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We investigate sandpile models where the updating of unstable columns is done according to a stochastic rule. We examine the effect of introducing nonlocal relaxation mechanisms. We find that the models self-organize into critical states that belong to three different universality classes. The models with local relaxation rules belong to a known universality class that is characterized by an avalanche exponent $\tau \approx 1.55$, whereas the models with nonlocal relaxation rules belong to new universality classes characterized by exponents $\tau \approx 1.35$ and $\tau \approx 1.63$. We discuss the values of the exponents in terms of scaling relations and a mapping of the sandpile models to interface models. [S1063-651X(97)03007-9]

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The concept of self-organized criticality (SOC), proposed in a seminal paper by Bak, Tang, and Wiesenfeld [1], has made considerable impact on a number of fields in the natural and social sciences. The paradigm of SOC is an idealized sandpile where grains added to a pile dissipate their potential energy through avalanches with no characteristic scale [1–9]. Early experimental studies of real sandpiles lead to clear disagreement with the numerical models: Bounded distributions of avalanche sizes were observed instead of the expected power-law dependence [10–14]. On the other hand, recent rice-pile experiments found power-law distributions of avalanche sizes [15] and tracer transit times [16]. These results sparked a renewed interest in the study of sandpile models [16–23].

A class of sandpile models with stochastic toppling rules, which is denoted ''rice-pile'' models, was found to display SOC in one dimension with a power-law distribution of avalanche sizes characterized by the critical exponent $\tau \approx 1.55$ [16–20]. Recently, Ref. [24] proposed a mapping of the model in [16] to the motion of a linear interface through a disordered medium [25]. For this universality class, to which we will refer to as the local linear interface (LLI) universality class, the mapping allows the determination of all the exponents characterizing the dynamics of the pile [24,26] and shows that τ is clearly different from the mean-field value $\tau = 3/2$ (see, e.g., [27]). Several other SOC models have been conjectured to be in the LLI class [23,24,28,29]. Thus the question arises of what mechanisms lead to the emergence of the LLI universality class for one-dimensional stochastic sandpile models with a preferred direction. Previously, sandpile models in higher dimensions have been classified according to their degree of ''directedness'' [2,30].

In this paper we undertake an investigation of the mechanisms responsible for the emergence of a given universality class for one-dimensional sandpile models with stochastic toppling rules and we discuss the reasons for the robustness of the LLI universality class. To this end, we study a class of models with stochastic toppling rules. We use as our basic model the one we proposed in Ref. [17] and investigate the robustness of the critical behavior upon modification of the toppling process. Specifically, we study stochastic variants of the models proposed in [2].

We find that the local models belong to the LLI universality class and have $\tau \approx 1.55$, while the nonlocal models belong to *new* universality classes with $\tau \approx 1.35$ and $\tau \approx 1.63$. In order to understand the new exponent values, we discuss scaling relations fulfilled by the exponents and the mapping to interface models.

First, we define the class of one-dimensional models: The system consists of a plate of length L, with a wall at $i=0$, and an open boundary at $i=L+1$. The profile of the pile evolves through two mechanisms: deposition and relaxation. Deposition is always done at $i=1$ and one grain at a time. During relaxation we look at all *active* columns of the pile: A column i of the pile is considered active if, in the previous time step, it (i) received a grain from column $i-1$, for the local models, or from $i-j$, $j=1,2,\ldots N$, for the nonlocal models, (ii) toppled a grain to column $i+1$, or (iii) column $i+1$ toppled one grain to its right neighbor. If a column i is active *and* the local slope $\delta h_i \equiv h(i)-h(i+1) > S_1$, then with probability $p(\delta h_i)$ several grains will be toppled from column i. Here we study the case

$$p(\delta h_i) = \min\{1, g(\delta h_i - S_1)\}, \tag{1}$$

where $g \leqslant 1$ is a parameter [31]. The number of grains N to be toppled is determined by either the *limited* (l) or *unlimited* (u) rule [2]:

$$N = \begin{cases} N_0, & (l) \\ \delta h_i - S_1, & (u). \end{cases} \tag{2}$$

The toppled grains are then redistributed according either to the *local* (L) or *nonlocal* (N) rule [2]

$$\begin{aligned} h(i+1) &= h(i+1) + N, \quad (L) \\ h(i+j) &= h(i+j) + 1, \quad j = 1, \ldots, N, \quad (N). \end{aligned} \tag{3}$$

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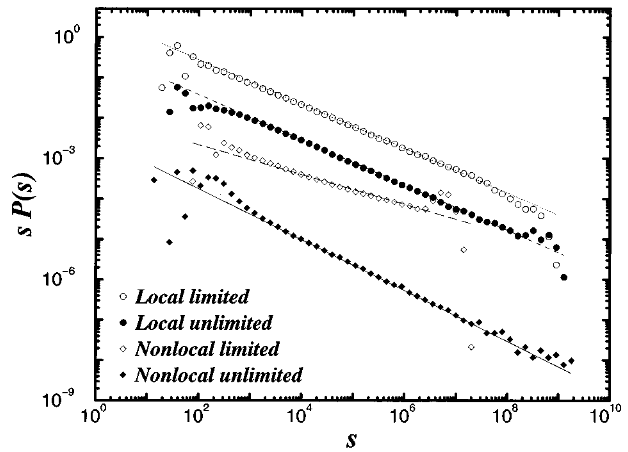

FIG. 1. A log-log plot of the probability density of the dissipated potential energy s during an avalanche. For greater clarity, the data for the Lu, Nl, and Nu were divided by factors of 10^1, 10^2, and 10^3, respectively. $P(s)$ was multiplied by s to make the change in the exponent τ visually clearer. For each model, we performed runs with the order of 10^7 grains deposited. The results shown are for $L=1600$, $g=1/8$, $S_1=6$, and $N_0=2$ (for the limited models). Other values of S_1 (ranging from 1 to 6), $1/g$ (ranging from 4 to 8), and N_0 (ranging from 2 to 4) were also investigated without any observable change in the estimate of the exponents. The slopes of the straight lines correspond to the exponent values from Table I. For the Nl model the bump before the cutoff leads to finite-size corrections, and we carried out simulations on systems up to size $L=20\,000$ in order to obtain more accurate estimates for the exponents.

Grains toppled to columns $i>L$ leave the system. We designate the models by the two letters that describe the rules applied: The local limited and unlimited models are referred to as Ll and Lu, while the nonlocal limited and unlimited models are referred to as Nl and Nu. The parameter $p(\delta h_i)$ describes friction between the grains and accounts for the fact that a large range of ''stable'' slopes was observed in the rice-pile experiment. The parameter g accounts for the effect of gravity on the packing configurations. For a discussion of the physical interpretation of the models see also Refs. [17,18].

We study the models in the slowly driven limit, i.e., we take the rate of deposition to be slow enough that any avalanche, that might be started by a deposited particle, will have ended before a new particle is deposited. The simulations of the models show that each system quickly enters a steady state which is characterized by varying avalanche sizes and a complex structure in time [17]. The size s of an avalanche can be defined in a number of ways. First, we follow the definition of Ref. [15] and calculate the size of an avalanche as the total potential energy dissipated in between deposited particles. In Fig. 1 we show the avalanche distributions for the various models. We find that the probability density function can be well described by the power-law form

$$P(s,L)=s^{-\tau}f(s/L^{\nu}), \tag{4}$$

where τ and ν are critical universal exponents. [Alternatively, we have that $P(s,L)=L^{-\beta}\tilde{f}(s/L^{\nu})$, where $\beta=\nu\tau$.]

TABLE I. Critical exponents for the four models studied. The definition of the critical exponents and classification of the models are given in the text. The data strongly suggest that the local models Ll and Lu belong to the LLI universality class, whereas the nonlocal models Un and Ln belong to new universality classes.

Model	τ	ν	y	σ
Ll	1.55 ± 0.02	2.24 ± 0.03	1.83 ± 0.04	1.42 ± 0.03
Lu	1.56 ± 0.02	2.26 ± 0.03	1.91 ± 0.04	1.37 ± 0.03
Nl	1.35 ± 0.05	1.55 ± 0.05	1.60 ± 0.05	0.95 ± 0.05
Nu	1.63 ± 0.02	2.75 ± 0.05	2.20 ± 0.04	1.50 ± 0.04

We used plots of consecutive slopes for different system sizes to obtain the estimates for τ (see Table I). Then these values were used to collapse the data according to Eq. (4) and extract ν.

The results from our numerical simulations show that the Ll and Lu models belong to the LLI universality class with $\tau\approx1.55$ and $\nu\approx2.25$, in agreement with the results of Refs. [16,17]. On the other hand, for the *nonlocal* models we obtain values of the exponents that signal the existence of new universality classes. The nonlocal limited (Nl) model is characterized by the exponents $\tau=1.35\pm0.05$ and $\nu=1.55\pm0.05$. The combined change of the number of toppled particles together with a nonlocal relaxation leads to yet a new universality class: For the nonlocal unlimited (Nu) model, we obtain the exponent values $\tau=1.63\pm0.02$ and $\nu=2.75\pm0.05$. We note that simple power laws, as in Eq. (4), in all cases provide us with nice data collapses. This result should be contrasted with the investigation of similar rules for the (nonstochastic) models in Ref. [2], where the results could not be described by a simple power law, but instead required a multifractal scaling form.

To further test our conclusion regarding the universality classes for the rice-pile models, we study different definitions of avalanche size: When using the total number of topplings, we find the same values for τ and ν as quoted above, which is due to the fact that a toppling event on average dissipates a fixed amount of potential energy (for a bounded distribution of slopes). For the lifetime T, we find that the probability density function is well described by the scaling form

$$D(T,L)=T^{-y}g(T/L^{\sigma}), \tag{5}$$

with the exponent values listed in Table I (see also Fig. 2). These results reassure us that the Ll and Lu models indeed belong to the LLI universality class, whereas the Nl and Nu models belong to new universality classes.

Next, we discuss scaling relations that are obeyed by the critical exponents. It is well established that the average avalanche size scales as

$$\langle s\rangle\sim L^{q}, \tag{6}$$

with the value of the exponent q depending on how the pile is driven. Here we have $q=1$ for all models (see, e.g., [2,23] for other q values). Combining Eqs. (4) and (6), we obtain the exponent relation [2]

$$\nu(2-\tau)=q, \tag{7}$$

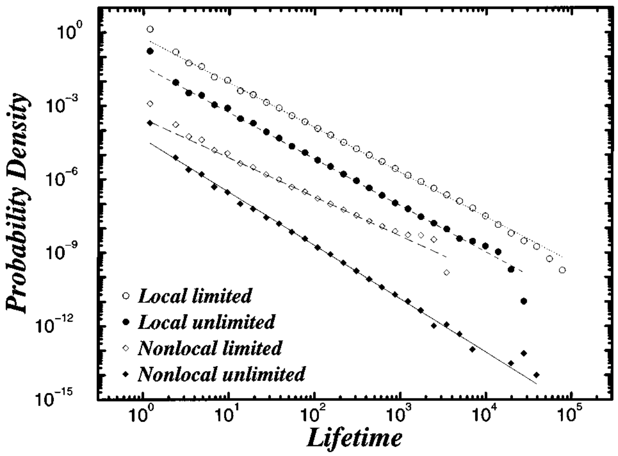

FIG. 2. A log-log plot of the probability density for the lifetime of avalanches. For greater clarity, the data for the Lu, Nl, and Nu were divided by factors of 10, 10^3, and 10^4, respectively. The same values were used for the parameters as in Fig. 1. The slopes of the straight lines correspond to the exponent values from Table I.

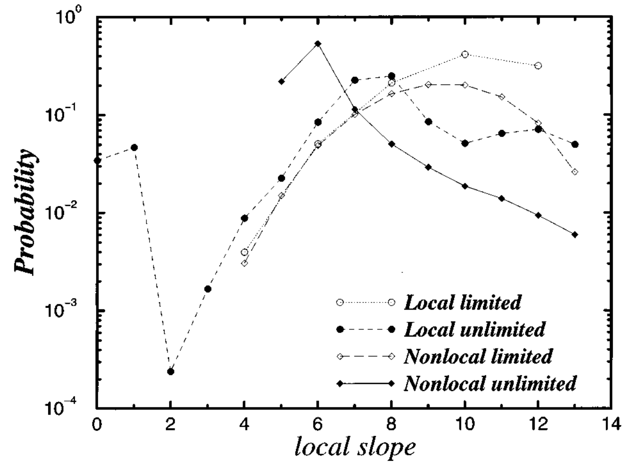

FIG. 3. A linear-log plot of the probability of the local slopes in the steady state. The parameters of the runs are the same as in Fig. 1, except that the system size is $L=400$. It is visually apparent that while the limited models lead to truncated Gaussian distributions, the unlimited models lead to more complicated forms. In particular, the Nu model leads to a nearly exponential decay of the probability of finding slopes larger than S_1 (which is related to the increase in the value of τ for this model).

which is fulfilled by all models. We have used $\langle s\rangle \sim L^{\nu(2-\tau)}$ (in combination with $\langle s^2\rangle$) to obtain alternative estimates of the critical exponents and we obtain values in complete accordance with those listed in Table I. Relations similar to Eqs. (6) and (7) can be derived for the lifetimes regarding the averages $\langle T\rangle,\langle T^2\rangle$ and then used to extract the critical exponents. Notice, however, that for the Nu model, $\langle T\rangle$ is a constant since we have $y>2$. From the scaling of the probability densities (4) and (5) we obtain

$$\nu(\tau-1)=\sigma(y-1), \tag{8}$$

in agreement with our results ($s\sim T^{\omega}$, with $\omega=\nu/\sigma$). The above relations imply that there are only two independent exponents.

As noted above, it was shown in Ref. [24] that the rice-pile model in [16] can be mapped to a linear interface model described by the continuum equation

$$\frac{\partial H}{\partial t}=D\frac{\partial^2 H}{\partial x^2}+\eta(x,H). \tag{9}$$

Here H, which is obtained as $h(x,t)=H(x-1,t)-H(x,t)$, counts the number of topplings of a given column and $\eta(x,H)$ is a quenched ''noise,'' which is related to the stochastic toppling probability. The rice-pile dynamics imposes a driving of the interface at $x=0$. The mapping predicts that $\nu=1+\chi$, where χ is the so-called roughness exponent characterizing, e.g., the scaling of the width of the interface with system size L (see, e.g., [32]). Numerically, one has $\chi\approx1.25$ [25], in excellent agreement with the value $\nu\approx2.25$. In addition, the cutoff exponent for the avalanche lifetime is $\sigma=z$, where z is the so-called dynamic exponent that describes the propagation of correlations.

The results presented here show that the Ll and Lu models belong to the LLI universality class. This result can be easily understood from the mapping to the linear interface model: The toppling of several particles does not change the fact that the growth of the interface is still local. Since the surface tension term is the relevant term (in the renormalization-group sense) it follows that the coarse-grained behavior of the local models is governed by the interface equation (9) as for the models in Refs. [16,17].

On the other hand, for the Nl and Nu models, the nonlocal toppling rules generate nonlocal growth that affects the interface motion. It has previously been shown that nonlocal interactions will in general lead to the emergence of new universality classes [33,34], which is confirmed by our observations for the Nl and Nu models. The critical exponents for the interface equations corresponding to the Nl and Nu models are not known. We can, nevertheless, qualitatively understand the change in the values of the exponent τ as follows: In general, we would expect that a nonlocal toppling rule would lead to a decrease of the value of τ because more columns in the pile are perturbed at every time step, thus creating larger avalanches. This is indeed what happens for the Nl model. However, for the Nu model we observe an increase in the value of τ. To understand this result, it is useful to look at the average slope of the pile and at the distribution of local slopes for the different models (see Fig. 3). For the limited models, the average slope remains practically the same upon the change of the number of particles toppled: 9.3 for the Ll model and 9.1 for the Nl model. This means that the change from one (local) to many (nonlocal) columns being made ''active'' works as we described above, i.e., it leads to a lower value of τ. On the other hand, for the unlimited models, a large change in the average slope is observed when we change the number of particles toppled: 7.8 for the Lu model and 6.2 for the Nu model. This implies that, on average, only a few particles are toppled from unstable columns for the Nu model, but for the Lu model more particles are toppled and this leads to a higher likelihood of big avalanches. As a result, we conclude that the Nu model should have a higher value of τ than the Lu model, as is indeed observed.

In summary, we study a class of rice-pile models that, in a simple way, model some of the physical features of the experiments in Refs. [15,16]. We find that for local relaxation rules the models belong to the LLI universality class that is characterized by the exponents $\tau \approx 1.55$ and $y \approx 1.87$ (models Ll and Lu). On the other hand, for models with nonlocal relaxation rules we obtain more complex dynamics resulting in new universality classes: For the Nl model we obtain $\tau \approx 1.35$ and $y \approx 1.60$, while for the Nu model we obtain $\tau \approx 1.63$ and $y \approx 2.20$. Our results show that nonlocal rules can increase or decrease the value of τ, thus opening the way for the possibility that the rice-pile experiment in Ref. [15] can be explained by some extension of nonlocal sandpile models by incorporating in detail the rice-grain dynamics.

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