Self-organized criticality in a rice-pile model

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We present a model for relaxations in piles of granular material. The relaxations are determined by a stochastic rule which models the effect of friction between the grains. We find power-law distributions for avalanche sizes and lifetimes characterized by the exponents $\tau=1.53\pm0.05$ and $y=1.84\pm0.05$, respectively. For the discharge events, we find a characteristic size that scales with the system size as L^{μ}, with $\mu=1.20\pm0.05$. We also find that the frequency of the discharge events decreases with the system size as $L^{-\mu'}$ with $\mu'=1.20\pm0.05$. [S1063-651X(96)51811-8]

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Since its introduction by Bak, Tang, and Wiesenfeld [1], the concept of self-organized criticality (SOC) and models which display SOC behavior have been the focus of much interest [1–9]. However, comparison with real systems has proved to be a tough test for the theory and models [10–13]. Furthermore, in one dimension the models tend to display either trivial behavior or behavior that cannot be classified as critical. Against this background, recent experiments on rice piles [14] have shown that under some conditions a real rice pile can self-organize into a critical state: For grains with a large aspect ratio the system self-organizes into a critical state. Frette *et al.* explained this result with the increased friction and packing possibilities that were able to cancel inertia effects. Furthermore, they observed that large local slopes developed in the pile.

Here, we propose a model for a pile of granular material where we introduce randomness in the relaxation rule instead of in the deposition rule. We study the model in one dimension and find power-law distributions for avalanche sizes s and lifetimes T. We also study the distribution of sizes for discharge events (i.e., particles falling off the pile), and find it to be bounded. The results show that our model belongs to a new universality class for systems displaying SOC.

First, we define the one-dimensional model. The system consists of a plate of length L, with a wall at $i=0$ and an open boundary at $i=L+1$. The profile of the pile evolves through two mechanisms: deposition and relaxation. Deposition is always done at $i=1$, and one grain at a time. The rate of deposition is slow enough that any avalanche, initiated by a deposited grain, will have ended before a new grain is deposited.

During relaxation we look at all *active* columns of the rice pile: A column i of the pile is considered active if, in the anterior time step, it (i) received a grain from column $i-1$, (ii) toppled a grain to column $i+1$, or (iii) column $i+1$ toppled one grain to its right neighbor. If a column i is active *and* the local slope, i.e., $\delta h(i)\equiv h(i)-h(i+1)$, is strictly larger than a threshold value S_1, then with probability p a grain will move from i to $i+1$. However, if $\delta h(i)>S_2$, a grain is moved from i to $i+1$ with probability 1. Grains toppled from column $i=L$ leave the system. When no active columns remain on the pile, the avalanche is said to be over.

The physical interpretation of our rules is the following: Suppose that a column, or portion, of the pile is in a metastable configuration. If a new grain is deposited or toppled on top of it, or the local slope changes, then that metastable configuration can become unstable. To model such an effect, we introduce the parameter p, which represents the fact that there is a finite probability that a new stable configuration is reached. Physically the parameter p thus describes the friction between the rice grains and the possibility that a metastable packing configuration will be attained. The friction effect in our model comes directly from the observation that there exists a large range of slopes in the rice pile instead of a single critical value [14]. The friction p can be a complicated function of local slopes and packings of the particles, but we find that the results are insensitive to the specific form and value of p. The parameter S_2 models the effect of gravity on the packing arrangements. We assume that above the maximum value S_2 of the local slope, it is no longer possible for a local stable configuration to be achieved, thus a grain must be toppled. In the limiting cases $p=0,1$, or $S_2=S_1$, we recover the model in Ref. [1] (which has trivial behavior for one dimension).

The simulation of the model shows two distinct regimes, a transient period followed by a steady (critical) state; cf. Fig. 1. Here, we focus on the properties of the model in the critical state. As can be seen in Fig. 1, the model leads to the establishment of a state with wildly varying avalanches sizes and a complicated structure in time. The size of an avalanche can be defined in a number of ways: the number of topplings s, the lifetime of the avalanche T, or the size of the discharge events m. We start by investigating the distribution of s. Figure 2(a) shows the probability density of avalanche sizes for different system sizes. The distribution follows the scaling form

$$P(s,L)\sim s^{-\tau}f_s(s/L^{\nu}), \qquad (1)$$

where f_s is a scaling function rapidly decaying for large arguments. The best collapse is obtained with the exponents $\tau=1.53\pm0.05$ and $\nu=2.20\pm0.05$, cf. Fig. 2(b). Even though τ is close to the mean-field value 3/2 [15–17], our model describes a different universality class. This can be shown by

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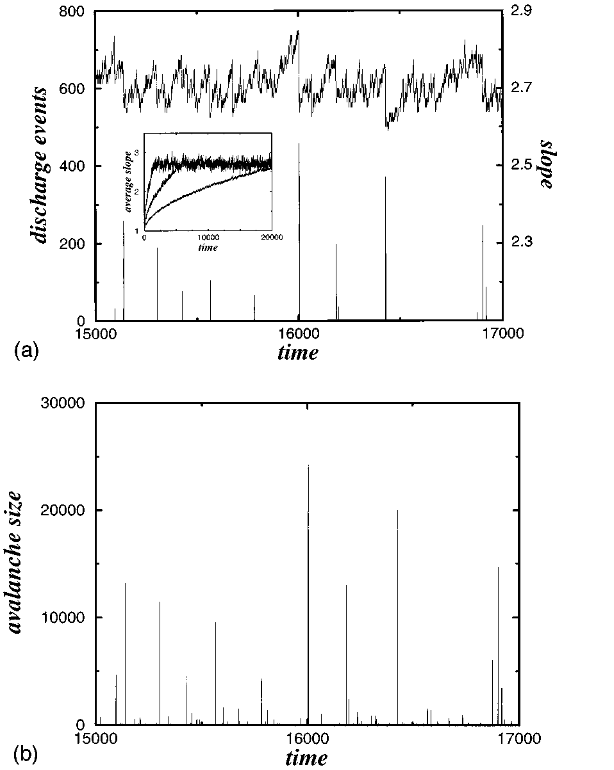

FIG. 1. (a) In the inset, we show the average slope as a function of time (measured as the number of deposited grains). As all the data presented in this paper, the average slope was obtained for $p=0.6$, $S_1=1$, and $S_2=4$. The system sizes shown are $L=40$, 80, 160. We can see that after an initial transient regime, whose duration depends strongly on the system size, a steady state is reached. The figure shows the fluctuations in the average slope of the pile, and the size of the discharge events, as a function of time for as system of size 80. It is visually apparent that there is a close connection between sharp changes in the average slope and the discharge events at the boundary. It is interesting to note that although our model does not include either a repose or a maximum angle for the pile, the dynamics *suggest* the existence of such angles because of the large discharge events. (b) Plot of the avalanches sizes as a function of time for the same system and interval as in (a). Again, the connection between the largest avalanches and the discharge events is observed.

mapping the avalanche dynamics to the motion of an interface through a disordered medium [18]. By using that the average number of topplings is $\langle s \rangle = L$ in the critical state, it follows from Eq. (1) that

$$\tau = 2 - \frac{1}{\nu}, \tag{2}$$

in agreement with our numerical results.

An interesting characteristic of the distribution is the presence of a peak, deviating from the power-law behavior, for a size close to the cutoff of the distribution. A close look at Fig. 1 shows that the biggest avalanches coincide with large changes in the average slope of the pile and with discharge events. Furthermore, as shown in Fig. 3, the number n_d of avalanches reaching the open boundary, for a given number of deposited grains, scales with the system size as

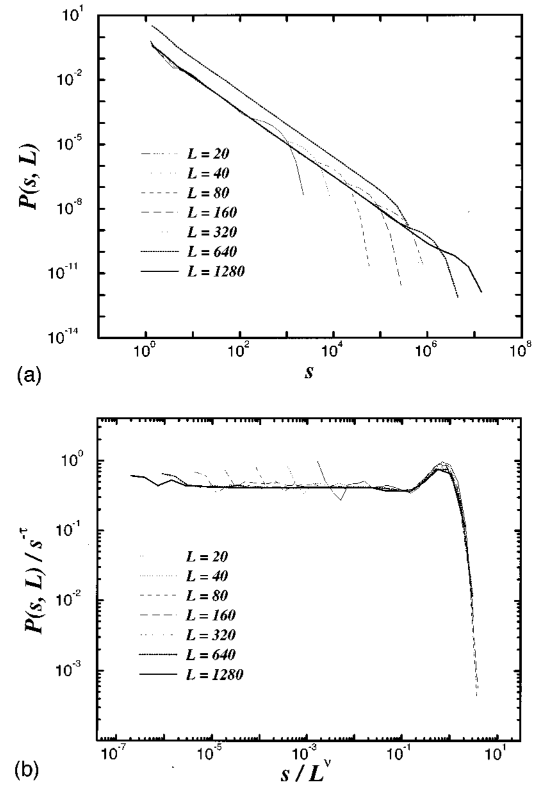

FIG. 2. (a) Log-log plot of the probability density of avalanche sizes s for several system sizes. It is visually apparent that for $s \gg 1$ a power-law dependence is observed. For values of s close to the cutoff, imposed by the finiteness of the system, we observe a peak deviating from the power-law behavior. To show that the peak is due to the system reaching a supercritical state (which is followed by a discharge) we also plot the distribution of avalanches when *no* discharge event occurred: That curve, obtained for $L=640$, is shifted vertically by a factor of 8, to make it more visible. (b) Data collapse of the curves shown in (a) according to Eq. (1) with the exponents $\tau \simeq 1.53$ and $\nu \simeq 2.20$.

$$n_d \sim L^{-\mu'}, \tag{3}$$

with $\mu' = 1.20 \pm 0.05$. This suggests that the peak is due to finite-size effects which lead the system into a supercritical

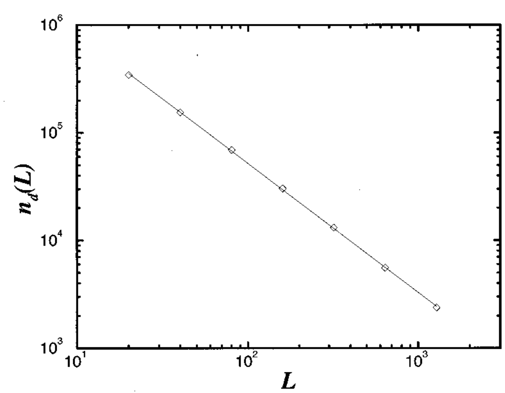

FIG. 3. The dependence of the number of discharge events n_d as a function of L. A power-law behavior, cf. Eq. (3), with $\mu' \simeq 1.20$ is obtained.

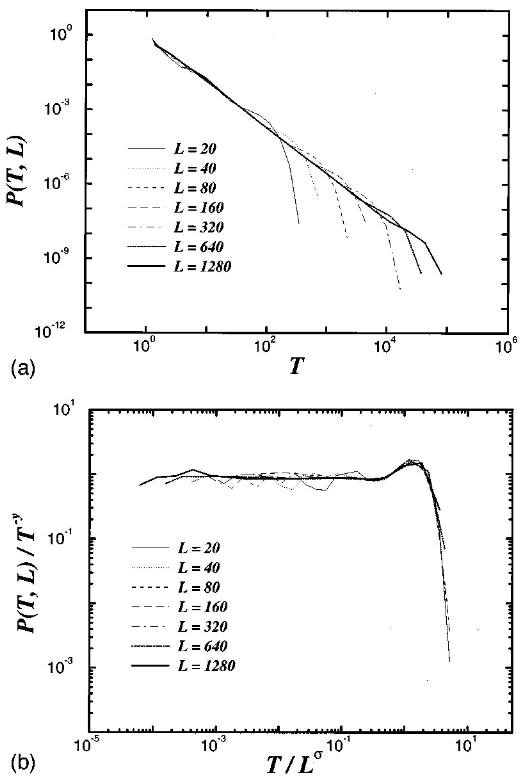

FIG. 4. (a) Log-log plot of the probability density of avalanche lifetimes T for several system sizes. It is visually apparent that for $t \gg 1$ a power-law dependence is observed. As for the avalanche sizes, a peak is present for lifetimes close to the cutoff. (b) Data collapse of the curves shown in (a) according to Eq. (4) with the exponents $y \simeq 1.84$ and $\sigma \simeq 1.40$.

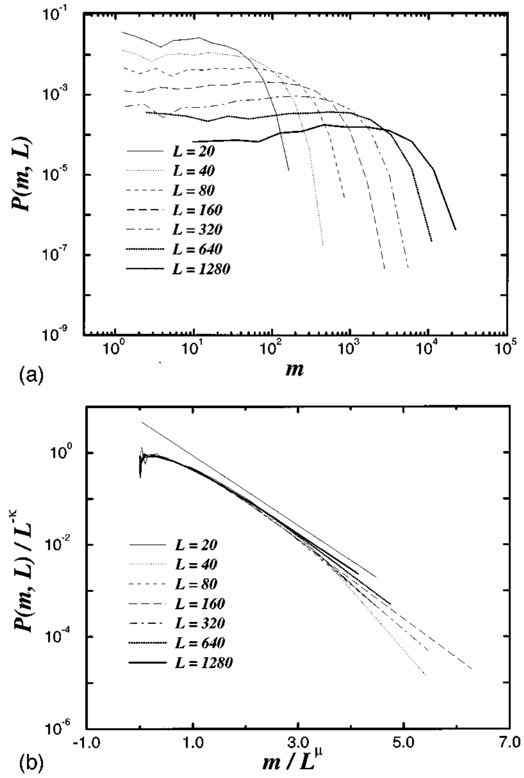

FIG. 5. (a) Log-log plot of the probability density of the sizes m of the discharge events for several system sizes. It is visually apparent that the distribution is bounded. (b) Data collapse of the curves shown in (a) according to Eq. (5) with the exponents $\kappa = \mu \simeq 1.20$. As a visual aid, we display a line corresponding to an exponential dependence.

state, followed by a massive avalanche and a large change in the average slope. We check this hypothesis by considering *only* the avalanches for which no discharge occurred at the boundary. As can be seen in Fig. 2(a), the peak for very large values of s is then no longer present and the cutoff has moved to a smaller value, confirming our hypothesis. We find that the data is described by Eq. (1) with the same values of the exponents as when the peak is present.

Next, we study the distribution of lifetimes T for the avalanches. As shown in Fig. 4(a), the data is described by the scaling form

$$P(T,L) \sim T^{-y} f_T(T/L^{\sigma}), \tag{4}$$

which is confirmed by the good data collapse obtained with the exponents $y = 1.84 \pm 0.05$ and $\sigma = 1.40 \pm 0.05$. From conservation of probability follows that $\sigma(y-1) = \nu(\tau - 1)$ [18], in nice agreement with our results. Finally, we study the distribution of sizes m for the discharge events [see Fig. 5(a)]. The scaling ansatz

$$P(m,L) \sim L^{-\kappa} f_m(m/L^{\mu}), \tag{5}$$

where the scaling function f_m decays exponentially, describes the data. Since the distribution $P(m,L)$ does not diverge for $m \to 0$ and its integral must equal 1, it follows that $\kappa = \mu$. This result is confirmed by the data collapse shown in Fig. 5(b), obtained for the exponents $\kappa = 1.2 \pm 0.1$ and $\mu = 1.2 \pm 0.1$.

It is possible to obtain additional scaling relations for the exponents besides those mentioned above. In the steady state, the input of matter must balance the output through the open boundary. Thus, we obtain that the frequency of discharge events must balance their characteristic size, and $\mu = \mu'$. The characteristic size of the discharge events depends on the system size as L^{μ}. So, we can conclude that whenever the system reaches a supercritical state, the number of grains discharged is of order L^{μ}. Since the average number of topplings for a given grain before being discharged is of order L, it follows that the cutoff size for the avalanches must scale as $L \times L^{\mu} \sim L^{\nu}$, thus $\nu = 1 + \mu$, in accordance with our results.

Just before submission of the present work, we became aware of a model by Christensen *et al.* [19], which for some range of parameters seems to belong to the same universality class as the model discussed here. The model in Ref. [19] introduces stochasticity in the toppling of particles via the selection of a new random critical slope for columns where a toppling occurred. For local slopes above the critical slope, a grain is always toppled. In Ref. [19] the predictions of the model are compared with experimental results for the diffusion of tracer particles. The numerical value of the exponent describing the diffusion of the tracers is in rough agreement with the numerical predictions of the model. The exponent

α describing the scaling of the potential energy dissipated during an avalanche is $\alpha \simeq 1.53$ for our model [20], in disagreement with the experimental result $\alpha \simeq 2$. In fact, the large disagreement between the experimental value of α and the numerical prediction of the model suggest that our model and the model of Ref. [19] do not belong to the same universality class as the rice-pile experiment. In [21], a stochastic sandpile model is studied in which only the front of the avalanches propagate (i.e., no backward avalanches are allowed). Such a rule leads to higher values for τ but apparently at the cost of destroying universality.

Frette *et al.* also found that for ''round'' rice grains the system did not evolve into a critical state, and that the distribution of avalanche sizes was bounded. The reason for this result can be understood if the results for the role of inertia on the dynamics of sandpiles are remembered [22,23]. It was shown in Ref. [22] that the assumption of zero inertia is essential for the establishment of the critical state. For real experiments, where inertia cannot be avoided, that assumption can only be valid for system with sizes smaller than a threshold value L_c [22,23]. Thus, for the round rice grains all system sizes studied in the experiments are larger than L_c, while for the elongated grains the opposite is true. Since our model has the implicit assumption of zero inertia it is inevitable that we will only be able to investigate the regime $L \ll L_c$, where SOC is observed.

In summary, we present a physically motivated model for piles of granular material. We find that the model self-organizes into a critical state with distributions for most quantities described by power laws. We measure the exponents characterizing these distributions, discuss scaling relations, and find that our model belongs to a new universality class.

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